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Supersymmetric embedding of the quantum hall matrix model

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Abstract: We develop a supersymmetric extension of the Susskind-Polychronakos matrix theory for the quantum Hall fluids. This is done by considering a system combining two sets of different particles and using both a component field method as well as world line superfields. Our construction yields a class of models for fractional quantum Hall systems with two phases U and D involving, respectively N_1 bosons and N_2 fermions. We build the corresponding supersymmetric matrix action, derive and solve the supersymmetric generalization of the Susskind-Polychronakos constraint equations. We show that the general U(N) gauge invariant solution for the ground state involves two configurations parameterized by the bosonic contribution k_1 (integer) and in addition a new degree of freedom k_2, which is restricted to 0 and 1. We study in detail the two particular values of k_2 and show that the classical (Susskind) filling factor ν receives no quantum correction. We conclude that the Polychronakos effect is exactly compensated by the opposite fermionic contributions.

Keywords: Chern-Simons Theories, Superspaces, Non-Commutative Geometry.

Contents

1. Introduction

In a work by Susskind [1] it was asserted that the non-commutative Chern-Simons (NCCS) gauge theory in (2+1) dimensional space is equivalent to Laughlin theory [2] at filling factor $\nu=\frac{1}{k}$, with k a positive (odd) integer. This formulation leads to a matrix model similar to that describing $D0$-branes in string theory and has opened a revival interest in the study of the fractional quantum Hall effect [3] in terms of NCCS theory. In particular a regularized version has been proposed by Polychronakos [4] where a quantum correction to the Susskind filling factor and the corresponding ground state has been constructed [5]. Other investigations about the relation between NCCS and Laughlin fluids can be found in recent works [6, 7, 8]. The Susskind model and its regularized version has been extended to quantum Hall (QH) states that are not of the Laughlin type in a multi-component Chern-Simons approach [9] and another based on the Haldane hierarchy [10] has been developed [11, 12]. Also a matrix model for bilayered QH systems [13] has been considered as well as an isotropic QH one [14].

In this paper we propose a supersymmetric matrix model (SMM) for quantum Hall fluids and thus generalized the original Susskind-Polychronakos (SP) theory for the Laughlin states. It can be used to describe a class of fractional QH systems resulting from two sets U and D of N_1 bosons and N_2 fermions. In our proposal, the states of these sets are imagined as two states of a world line supersymmetric representation with $N_1 = N_e$ and $N_2 = N_{\widetilde{e}}$ where $\widetilde{e}$ is the superpartner of the particle e. Our supersymmetric extension (SE) of the SP model gives new solutions for Laughlin states with the total filling factor $1/k$. More precisely, we consider a QH system of N bosons and fermions ($N = N_1 + N_2$) with $N_\phi = kN$ quantum fluxes ($N_\phi = k_1 N_1 + k_2 N_2$) and investigate its basic features.

To achieve this proposal, we develop a SMM for a fractional QH system and show how this can describe U and D together. We write the SMM action and derive the supersymmetric constraint equations. These will be solved to obtain wave functions for the lowest Landau level and the corresponding filling factors. Throughout this study, we obtain new results which can be interpreted in terms of supersymmmetric representation theory. In particular, we find two Polychronakos effects coming with opposite signs and cancel each other exactly in agreement with Bose-Fermi cancellation.

Before doing this task, we develop and discuss another interesting result. This concerns a general relation governing filling factor and energy of a given QH system. It allows us to study the $1/N$ corrections in fractional QH systems and therefore give some information on the behavior of the collective motions of the interacting QH particles. This has another consequence, mainly it can be used to derive an immediate description of a QH system combining bosons and fermions.

The presentation of this paper is as follows. In section 2, we revisit the main steps of the SP construction. A link between filling factor ν and the energy will be given as well as a $1/N$ expansion of ν. This will be used in dealing with a QH system combining bosons and fermions from the quantum mechanics point of view. In section 3, we study the supersymmetric extension of the SP construction by using two approaches: component fields and world line superfields.[1] By distinguishing two different configurations, we build the ground states and determine their filling factors. We summarize our results in the last section.

2. Susskind-Polychronakos matrix model

We revisit the SP model by using a method of constrained gauge systems that we extend to the matrix quantum mechanics. It allows us to give another way to obtain the filling factor of the system as well as its $1/N$ expansion. Of course for a large number of particles N such $O(1/N)$ corrections are negligible. On the other hand, this approach can also be applied to the SE of the SP model, which we will develop in the forthcoming section.

[1] A general treatment of $\mathcal{N} = 1$ supersymmetry (SUSY) in four-dimensional can be found in [15]. However, a systematic development of the representation theory of supersymmetric 1 dimensional, arbitrarily $\mathcal{N}$-extended SUSY can be found in [16].

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To start recall that the SP action $S[X^a, \Psi, A]$, describing a system of N electrons in the presence of a strong magnetic field B, is given by

$$S\left[X^a, \Psi, A_0\right] = -\frac{B}{2}\int dt\, \varepsilon_{ab}\,\mathrm{Tr}\left[\left(\dot{X}^a - i\left[A, X^a\right]\right)X^b + \theta\varepsilon^{ab}A_0\right] + \\ + \int dt \left\{\sum_{n,m=1}^{N} i\Psi_n^*\left(\delta_{nm}\partial_t - iA_{nm}\right)\Psi_m - \frac{B\omega}{4}\sum_a \mathrm{Tr}\left(X^a\right)^2\right\} \tag{2.1}$$

where θ is the non-commutativity parameter, $X^a(t)$ and $A_0 = A(t)$ are one-dimensional (1D) $N \times N$ hermitian matrices but Ψ is the Polychronakos field in the fundamental of $\mathrm{U}(N)$. Since $X^a(t)$ and $A(t)$ are hermitian, we may attempt to diagonalize them by using $\mathrm{U}(N)$ invariance of the action and work directly with the field eigenvalues. This way seems easy and interesting in quantizing the system, but we will not follow this path. Indeed, we will keep $X^a(t)$ and $A(t)$ as matrices and impose gauge invariance on the Hilbert space as usual in constrained systems.

The equations of motion for X^a and Ψ, respectively, are

$$\begin{aligned} \varepsilon_{ab}\nabla_t X^b + \omega X^a &= 0 \\ \nabla_t \Psi_n &= 0 \end{aligned} \tag{2.2}$$

where $\nabla_t = \partial_t - iA$, There is also an additional equation of motion for A. Since this latter is one of the key object in this analysis, we discuss it in detail below.

2.1 Constraint equation

The gauge field A has no kinetic term and can be eliminated via its equation of motion. This gives the Gauss constraint

$$i\frac{B}{2}\varepsilon_{ab}[X^a, X^b]_{nm} + \Psi_n^*\Psi_m = B\theta\delta_{nm}\,. \tag{2.3}$$

Since the hermitian matrices X^a are in the $\mathrm{U}(N)$ algebra, it is convenient to rewrite (2.3) in terms of the $\mathrm{U}(N) = \mathrm{U}(1) \oplus \mathrm{SU}(N)$ generators $\mathcal{J}_{nm}$

$$\mathcal{J}_{nm} = B\theta\delta_{nm}\,, \tag{2.4}$$

on the other hand

$$\mathcal{J}_{nm} = i\frac{B}{2}\varepsilon_{ab}\left[X^a, X^b\right]_{nm} + \Psi_n^*\Psi_m\,. \tag{2.5}$$

Upon setting

$$Z^{\pm} = \frac{1}{\sqrt{B}}\left(X^1 \pm iX^2\right) \tag{2.6}$$

it is easily seen that (2.5) takes the form

$$\mathcal{J}_{nm} = \left[Z^+, Z^-\right]_{nm} + \Psi_n^*\Psi_m \tag{2.7}$$

where $Z = Z^+$ and $\overline{Z} = Z^-$. Using the $\mathrm{U}(N)$ generator basis $\{T^I, 1 \le I \le N^2 - 1, T^{N^2} \equiv Q\}$, with Q is the conserved charge, satisfying the usual commutation relations

$$\left[T^I, T^J\right] = if_K^{IJ}T^K \tag{2.8}$$

and expanding Z as

$$Z = \sum_I Z_I T^I \tag{2.9}$$

we can show that (2.7) can be written as

$$\mathcal{J}_{nm} = \sum_{K=1}^{N^2-1} \mathcal{J}_{nm}^K + \mathcal{J}_{nm}^0 \tag{2.10}$$

where the components $\mathcal{J}_{nm}^K$ are given by

$$\begin{aligned} \mathcal{J}_{nm}^K &= \left(i\overline{Z}_I Z_J f_K^{IJ} + \Psi^* T^K \Psi\right) T_{nm}^K \\ \mathcal{J}_{nm}^0 &= \left(\Psi^* Q \Psi\right) Q_{nm} \,. \end{aligned} \tag{2.11}$$

Since the first $N^2 - 1$ generators are traceless and

$$\mathrm{Tr}(Q) = N \tag{2.12}$$

(2.11) can be simplified as

$$\begin{aligned} \mathcal{J}_{nm}^K &= 0 \\ \mathcal{J}_{nm}^0 &= B\theta\delta_{nm} \,. \end{aligned} \tag{2.13}$$

This can be seen as just the classical conditions for the SU(N) gauge invariance and the fixed total U(1) charge

$$q = \mathrm{Tr}\left(\mathcal{J}_{nm}^0\right) = B\theta N \,. \tag{2.14}$$

2.2 Quantum analysis

The study of constrained gauge systems teaches us that quantum mechanically the above constrained relations should be written as

$$\langle \mathcal{F} | \mathcal{J}_{nm}^K | \mathcal{F} \rangle = 0 \tag{2.15}$$

$$\langle \mathcal{F} | \,\mathrm{Tr}\, \mathcal{J}^0 | \mathcal{F} \rangle = B\theta N \tag{2.16}$$

where $|\mathcal{F}\rangle$ are the wave functions of the underlying Hilbert space $\mathcal{H}$ of the system. The first equation tells us that $|\mathcal{F}\rangle$ should satisfy

$$T_{nm}^K |\mathcal{F}\rangle = 0 \tag{2.17}$$

and reflects just the SU(N) gauge invariance. Quantization of the U(N) charges requires that

$$B\theta = kN \,, \qquad k \in \mathbb{Z}_+^* \,. \tag{2.18}$$

The second constraint (2.16) implies that $|\mathcal{F}\rangle$ should carry kN U(1) charges.

Furthermore in quantum mechanics with matrix variables, the usual classical conjugate momentum Π_ϕ and the variables ϕ satisfying the Poisson brackets

$$\{\Pi_\phi, \phi\}_{\mathrm{Poisson}} = 1 \tag{2.19}$$

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are roughly speaking interpreted as creation $\mathbf{O}^\dagger$ and annihilation $\mathbf{O}$ operators obeying generalized Heisenberg commutation relations. For the case of the square matrix fields X and the Polychronakos vector variable Ψ, we have the relations

$$\begin{aligned} \left[\mathbf{A}^\dagger_{nm}, \mathbf{A}^-_{ij}\right] &= \delta_{mi}\delta_{nj} \\ \left[\mathbf{\Psi}^\dagger_i, \mathbf{\Psi}^-_j\right] &= \delta_{ij}\,. \end{aligned} \tag{2.20}$$

The hamiltonian $\mathbf{H}$ of the first quantized matrix model is

$$\mathbf{H} = \omega\left(\mathcal{N}_{\mathrm{A}} + \frac{a_{\mathrm{b}}}{2}\right) \tag{2.21}$$

where $\mathcal{N}_{\mathrm{A}}$ is the matrix operator number

$$\mathcal{N}_{\mathrm{A}} = \sum_{n,m} \mathbf{A}^\dagger_{nm}\mathbf{A}^-_{nm} \tag{2.22}$$

and the extra term $a_{\mathrm{b}}/2 = N^2/2$ is due to quantum corrections generated by the central term of (2.20). As we noted previously, in this matrix quantum mechanical treatment, the result of (2.5) is now replaced by a constraint on the wave functions $|\mathcal{F}\rangle$ of $\mathcal{H}$:

$$J_{nm}|\mathcal{F}\rangle = 0 \tag{2.23}$$

$$J^0|\mathcal{F}\rangle = kN|\mathcal{F}\rangle\,. \tag{2.24}$$

$|\mathcal{F}\rangle$ are invariant under $\mathrm{SU}(N)$ and carry a well defined J^0 charge. To obtain the solution of these constraint equations, note that

$$\begin{aligned} \left[J^0, \mathbf{\Psi}^\dagger\right] &= \pm\mathbf{\Psi}^\dagger \\ \left[J^0, \mathbf{\Psi}^\dagger\mathbf{\Psi}^-\right] &= 0 \end{aligned} \tag{2.25}$$

which tell us that the $\mathbf{\Psi}^\dagger$ and $\mathbf{\Psi}^-$ dependence in $|\mathcal{F}\rangle$ should be of the form $(\mathbf{\Psi}^\dagger_i)^{kN}$ times an arbitrary function of $\mathbf{\Psi}^\dagger\mathbf{\Psi}^-$, that is $(\mathbf{\Psi}^\dagger)^{kN} \times f(\mathbf{\Psi}^\dagger\mathbf{\Psi}^-)$. The solution of condition (2.23) leads to the following normalized ground state

$$|\mathcal{F}_0\rangle = \mathcal{N}\left[\varepsilon^{\alpha_1\cdots\alpha_N}\,\mathbf{\Psi}^\dagger_{\alpha_1}\left(\mathbf{\Psi}^\dagger\cdot\mathbf{A}^\dagger\right)_{\alpha_2}\cdots\left(\mathbf{\Psi}^\dagger\cdot\mathbf{A}^{\dagger N}\right)_{\alpha_N}\right]^k|0\rangle\,. \tag{2.26}$$

One of the remarkable properties of (2.26) is that its fundamental energy E_0

$$E_0 = \frac{\langle\mathcal{F}_0|\mathbf{H}|\mathcal{F}_0\rangle}{\langle\mathcal{F}_0|\mathcal{F}_0\rangle} \tag{2.27}$$

behaves for large-N as

$$\frac{2E_0}{\omega} \sim N^2\,(k+1)\,. \tag{2.28}$$

Other remarkable features of this solution were considered in [5].

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2.3 Quantum flux

This physical quantity is relevant in determining the filling factor for the SP model, which is defined as the ratio between the number of particles N and the quantum flux number N_ϕ

$$\nu = \frac{N}{N_\phi} \,. \tag{2.29}$$

To make contact between this relation and the matrix model formalism, it is interesting to develop the calculus of N_ϕ in terms of the matrix operators of the Hall droplet. This can be done by considering the classical integral measure

$$d^2 N_\phi \equiv n_\phi = B dx^1 dx^2 \tag{2.30}$$

as the mean value over the N measure integrals $B dx_i^1 dx_i^2$ associated with the coordinates x_i^a within the droplet

$$B dx^1 dx^2 = \frac{B}{N} \sum_{i=1}^{N} dx_i^1 dx_i^2 = \frac{1}{N} \sum_{i=1}^{N} dx_i d\pi_i \tag{2.31}$$

where x_i and π_i stand for x_i^1 and $B x_i^2$ respectively. Now using the fact that x_i^a variables are nothing but the eigenvalues of the hermitian matrices X and Π, one can rewrite (2.30) as

$$n_\phi = \frac{1}{2N} \operatorname{Tr} \left(dX d\Pi + d\Pi dX \right) \tag{2.32}$$

where X ($X = X^1$) and its conjugate momentum Π ($\Pi = BX^2$) are representants of the $\mathrm{U}(N)$ gauge orbit

$$X \equiv U X U^\dagger \,, \qquad \Pi \equiv U \Pi U^\dagger \,, \qquad U \in \mathrm{U}(N) \,. \tag{2.33}$$

With these matrix variables, the classical relation for the flux integral

$$N_\phi = \int_{\text{droplet}} B dx^1 dx^2 \tag{2.34}$$

can be written as

$$N_\phi = \int_{\text{droplet}} \frac{1}{N} \operatorname{Tr} \left(dX d\Pi \right) = \frac{1}{2N} \operatorname{Tr}_{\mathcal{H}_{\text{droplet}}} \left(X\Pi + \Pi X \right) \tag{2.35}$$

where $\mathcal{H}_{\text{droplet}}$ stands for the Hilbert space of the droplet. This is a remarkable relation first because it expresses the number N_ϕ in terms of the coordinates of the phase space of the SP model and second because of its interpretation at the quantum level. By using an analysis similar to that we have adopted for (2.5), we can also obtain

$$N_\phi = \frac{1}{N} \sum_{|\mathcal{F}\rangle \in \mathcal{H}} \langle \mathcal{F} | \mathcal{N}_\phi | \mathcal{F} \rangle \tag{2.36}$$

where the flux number operator $\mathcal{N}_\phi$ that follows from (2.35) is given by

$$\mathcal{N}_\phi = \frac{1}{2} \sum_{n,m} \left(\mathbf{A}_{nm}^\dagger \mathbf{A}_{nm}^- + \mathbf{A}_{nm}^- \mathbf{A}_{nm}^\dagger \right) . \tag{2.37}$$

Now using (2.20) and (2.22), we can show that

$$\mathcal{N}_\phi = \mathcal{N}_{\rm A} + \frac{N^2}{2} \tag{2.38}$$

which implies in turn that for the SP model there is a close relation between $\mathcal{N}_\phi$ and $\mathbf{H}$, namely

$$\mathbf{H} = \omega \mathcal{N}_\phi \tag{2.39}$$

or equivalently by using (2.36)

$$E = \frac{\omega}{2} N_\phi N \,. \tag{2.40}$$

This tells us that the energy spectrum of the present system can be measured in terms of the quantum flux number and the number of electrons. Note also that for a fixed number of particles N, energy variation is induced by the variation of the flux.

2.4 $\mathcal{O}(1/N)$ expansion of the filling factor

Equation (2.40) is an interesting result since it has three different consequences. First, it relates the filling factor ν (2.29) to the energy of the states. Indeed, substituting

$$N_\phi = \frac{2E}{\omega N} \tag{2.41}$$

we end up with

$$\nu = \frac{\omega N^2}{2E} \,. \tag{2.42}$$

For the lowest Landau level of energy E_0, the quantum flux is

$$N_\phi \sim \frac{2E_0}{\omega N} \tag{2.43}$$

and ν reduces to the familiar formula

$$\nu_0 = \frac{\omega N^2}{2E_0} \,. \tag{2.44}$$

At this stage, we believe a remark is in order. As we know, it is not so easy to solve interacting QH systems. Nevertheless, one can always use some other technique to acquire more information about such systems. For instance, in the perturbation theory the energy E can be expanded as

$$E = E_0 \left(1 + \frac{E_1}{E_0} + \frac{E_2}{E_0} + \cdots \right) \tag{2.45}$$

where E_i are the perturbated energies and E_i/E_0 are smaller than one. Now (2.42) becomes

$$\nu = \frac{\omega N^2}{2E_0} \left(1 + \frac{E_1}{E_0} + \frac{E_2}{E_0} + \cdots \right)^{-1} \tag{2.46}$$

which suggests that one may expand it in a perturbative series

$$\nu = \sum_{n \geq 0} \nu_n \tag{2.47}$$

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with the leading term given by ν_0 of the ground state. For a strongly correlated system with a large gap, particles are mainly confined in the lowest Landau level. In this case, by using (2.39), we can show that (2.36) reduces to

$$N_\phi \sim \frac{1}{N}\langle \mathcal{F}_0|\mathcal{N}_{\rm A}|\mathcal{F}_0\rangle + \frac{N}{2}\,. \tag{2.48}$$

With the help of

$$\mathcal{N}_{\rm A}|\mathcal{F}_0\rangle = \frac{kN}{2}\,(N+1)\,|\mathcal{F}_0\rangle \tag{2.49}$$

which can be seen from (2.26), we obtain

$$N_\phi \sim k + N\,(k+1)\,. \tag{2.50}$$

This leads to the well-known filling factor

$$\nu_0 = \frac{1}{k+1} \tag{2.51}$$

which is in agreement with [4, 5].

A second comment concerns the general expression for E and N_ϕ. From the way of building up the SP wave functions $|\mathcal{F}\rangle$, we note that the expectation value

$$\langle \mathcal{F}|\mathcal{N}_{\rm A}|\mathcal{F}\rangle \tag{2.52}$$

can be usually written as a power series in terms of the integer N. This implies that the total energy E of the SP approach can be expanded as

$$E = \frac{\omega}{2}\sum_{s\geq 1} r_s N^s \tag{2.53}$$

where r_s are rational numbers whose two leading terms are

$$r_1 = k\,, \qquad r_2 = k+1\,. \tag{2.54}$$

Now by comparing (2.40) and (2.53), we obtain

$$N_\phi = \sum_{s\geq 1} r_s N^{s-1} = \frac{1}{N}\left(r_1 N + r_2 N^2 + r_3 N^3 + \cdots\right). \tag{2.55}$$

With this relation, one can compute the finite part of the ratio N_ϕ/N to obtain the general form of the filling factor in such way that

$$\nu = \frac{1}{k+1} - \frac{k}{N(k+1)} - \left(\frac{k}{N(k+1)}\right)^2 + \mathcal{O}\left(\frac{1}{N^3}\right) \tag{2.56}$$

which is a $1/N$ expansion series. Note that it converges to ν_0 (2.51) in the limit $N \to \infty$. The extra terms can be interpreted as $1/N$ quantum fluctuations.

A third aspect upon which we wish to comment concerns another property of (2.42). In general, this relation may be applied to the SE of the SP model, which we will study later. In the absence of mutual interactions between bosons and fermions, the total energy E of the supersymmetric QH system is the sum of two parts

$$E_{\mathrm{bf}} = E_{\mathrm{b}} + E_{\mathrm{f}} \tag{2.57}$$

where b and f refer, respectively, to bosons and fermions. Then by adopting the above analysis one expects that the corresponding filling factor is given by

$$\nu_{\mathrm{bf}} = \frac{\omega N^2}{2(E_{\mathrm{b}} + E_{\mathrm{f}})} \,. \tag{2.58}$$

3. Supersymmetric matrix model

Generally speaking a particle confined to a plane in the presence of a strong perpendicular magnetic field B behaves as a 1D field $x(t)$ ($\equiv x^1$) with conjugate momentum[2] $y(t)$ ($\equiv x^2$). The theories of such configurations are quite well understood and the matrix model of the previous section is one of them. For a system combining bosons and fermions, one may have an interesting situation that deserves to take care of. A generic configuration of such a system can be imagined as consisting of two phases: U involving N_1 bosons and D involving N_2 fermions. To fix the ideas, we take these numbers as

$$N_1 = k_1 N \,, \qquad N_2 = k_2 N \tag{3.1}$$

where k_1 and k_2 are positive integers. Note that for $k_2 = 0$, we have no D phase and the QH system is the one described in previous section. For non-zero k_2, the situation is important in the sense that is quite realistic. In particular, $k_2 = 1$ is interesting because it has much to do with the SE of the SP model. Therefore, we claim that our supersymmetric generalization of the SP theory modelled the U and D configurations with a positive odd integer k_1 and $k_2 = 1$.

3.1 Generalization to supersymmetric matrix theory

We start by recalling that in the SMM we have essentially the same tools and machinery as those we generally have in the SP model. The main difference is that the usual QH system is now replaced by another system describing multiplets involving bosonic and fermionic states. The main novelties in the SMM are of two types:

(i) This model describes two kinds of the fractional QH subsystems: bosons and fermions (superpartner) forming, respectively, U and D phases. Classically, they are described by two pairs of 1D fields namely the super-position coordinates

$$x_i^1(t) \,, \qquad \eta_i^1(t) \tag{3.2}$$

[2]For a system of N particles, the conjugate momenta $x_i^2(t)$ are thought of as the eigenvalues of a $N \times N$ hermitian matrix field $X^2(t)$. The X^1 and X^2 matrices satisfy the Heisenberg algebra with central extension proportional to the inverse of the external magnetic field.

interchanged under supersymmetric transformations (ST) and their respective conjugate momenta

$$x_i^2(t)\,, \qquad \eta_i^2(t)\,. \tag{3.3}$$

They are related by $1D$ supersymmetry (SUSY) as

$$\delta_\epsilon \eta^a = \epsilon x^a\,, \qquad \delta_\epsilon x^a = -i\epsilon\partial_t \eta^a \tag{3.4}$$

where ϵ is a parameter of the SUSY group with $\epsilon^2 = 0$ and $a = 1, 2$. Note that the ST we are using here is the world line SUSY. It should be distinguished from the target space SUSY which might also be considered for building more general super QH systems with rich invariance. We will comment later on this issue, but for the moment we will fix our attention mainly on our world line SMM.

(ii) In the SMM formulation, the multiplets (3.2) and (3.3) are replaced by a pair of 1D $N \times N$ hermitian matrix fields $X^a = X^a(t)$ and $\eta^a = \eta^a(t)$

$$X^a = \begin{pmatrix} X_{11}^a & .\,. & X_{1N}^a \\ . & .\,. & . \\ . & .\,. & . \\ X_{N1}^a & .\,. & X_{NN}^a \end{pmatrix}, \qquad \eta^a = \begin{pmatrix} \eta_{11}^a & .\,. & \eta_{1N}^a \\ . & .\,. & . \\ . & .\,. & . \\ \eta_{N1}^a & .\,. & \eta_{NN}^a \end{pmatrix}. \tag{3.5}$$

Since these matrices are hermitian, they can be diagonalized by making use of the $\mathrm{U}(N)$ transformations in such way that

$$\begin{aligned} V_{ik} X_{kl}^a V_{kj}^{-1} &= X_i^a \delta_{ij} \\ U_{ik} \eta_{kl}^a U_{kj}^{-1} &= \eta_i^a \delta_{ij}\,. \end{aligned} \tag{3.6}$$

As in the bosonic case, we will not fix such a symmetry at the level of the dynamical matrix field variables. We will use it as a basic symmetry in the action and only impose it at the level of the Hilbert space of wave functions describing the fractional QH system. Since we are going to consider large but still finite $N \times N$ matrices, we need to introduce the Polychronakos vector field $\Psi = \Psi(t)$ as well as its superpartner $\upsilon = \upsilon(t)$

$$\Psi = \Big(\Psi_1, \ldots, \Psi_N\Big), \qquad \upsilon = \Big(\upsilon_1, \ldots, \upsilon_N\Big). \tag{3.7}$$

To write the action of the world line SMM, we may use two different but equivalent ways: component field and world line superfield approaches. The latter involves auxiliary fields and is more general. For completeness, we shall consider in this paper both of them.

3.2 Component field method

One way to build the world line SMM is to use a direct method based on component field techniques. This allows us to derive immediately the world line SE of the constraint equation (2.3) and write down the corresponding supersymmetric invariant action. This can be done by considering the following ST for matrix variables[3]

$$\begin{aligned} \delta_\epsilon \eta^a &= \epsilon X^a\,, & \delta_\epsilon X^a &= -i\epsilon\partial_t\eta^a \\ \delta_\epsilon \upsilon &= \epsilon\Psi\,, & \delta_\epsilon \Psi &= -i\epsilon\partial_t\upsilon\,. \end{aligned} \tag{3.8}$$

They extend (3.4) and satisfy the following $1D$ $\mathcal{N}=1$ supersymmetric algebra

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,\phi = 2i\epsilon_1\epsilon_2\partial_t\phi \tag{3.9}$$

where ϕ stands for X^a, η^a, Ψ and υ. Before going on, let us list the contents of supersymmetric multiplets, which will be used. These are of two kinds:

(i) N^2 Susskind supermultiplets $(0^{N^2}, (1/2)^{N^2})$ described by (X_{ij}, η_{ij}) matrix field transforming in the $\mathrm{U}(N)$ adjoint representation.

(ii) N supermultiplets $((-1/2)^N, 0^N)$, $(-1/2)^N$ for the Polychronakos superpartner, described by (Ψ_i, υ_i) transforming as the vector of $\mathrm{U}(N)$. Since X^a and η^a are hermitian, they can be diagonalized as $(X_i\delta_{ij}, \eta_i\delta_{ij})$ and so one has a N-dimensional target space geometry. Otherwise the dimension would be N^2 due to the equivalence (3.6).

3.2.1 Supersymmetric constraint equations

We propose to study the SE of the constraint equations by considering classical and quantum analysis.

Classical constraints: from the above transformations together with the gauge condition $\delta_\epsilon A = 0$ (which in a complete supersymmetric formulation should involve a supersymmetric partner: gauginos) one can build the supersymmetric generalization of the SP theory just by demanding closure under (3.8). According to (2.2), we find the X and Ψ superpartner equations of motion

$$\begin{aligned} &\varepsilon_{ab}\nabla_t^2\eta^b + \omega\nabla_t\eta^a = 0 \\ &\nabla_t^2\upsilon_n = 0. \end{aligned} \tag{3.10}$$

Assuming that these equations result from the variation of an action $S_{\text{susy}} = S[\phi]$, i.e.

$$\delta S/\delta\phi = 0 \tag{3.11}$$

one can derive the SE of the SP action (2.1). Doing this integration calculus, we end up with

$$\begin{aligned} S_{\text{susy}} = &-\frac{B}{2}\int dt\varepsilon_{ab}\,\mathrm{Tr}\left(\nabla_t X^a X^b + \frac{i}{2}\nabla_t\eta^a\nabla_t\eta^b + \theta\varepsilon^{ab}A_0\right) \\ &+\int dt\,\left\{i\Psi^*\nabla_t\Psi + \nabla_t\upsilon^*\nabla_t\upsilon - \frac{B\omega}{4}\sum_a \mathrm{Tr}\left(X^aX^a - i\eta^a\nabla_t\eta^a\right)\right\}. \end{aligned} \tag{3.12}$$

[3] In presence of gauge fields, ∂_t should be replaced by $\nabla_t = \partial_t - iA_t$.

We can show that S_{susy} has the U(N) gauge invariance of the bosonic SP action. Actually we have two conserved currents $\mathcal{J}_{mn}$ and $\mathcal{F}_{mn}$ valued in U(N). They, respectively, generate bosonic and fermionic symmetries and combine to form a unique supercurrent in superspace formalism. $\mathcal{J}_{mn}$ has two contributions: bosonic $\mathcal{J}_{(\mathrm{b})mn}$ and fermionic $\mathcal{J}_{(\mathrm{f})mn}$ and can be expanded in the $u(N)$ basis $\{T^K, 1 \leq K \leq N^2\}$ as

$$\mathcal{J}_{mn} = \mathcal{J}_{(\mathrm{b})mn} + \mathcal{J}_{(\mathrm{f})mn} = \sum_{K=1}^{N^2} T^K \mathcal{J}_{nm}^K. \tag{3.13}$$

In a similar way we can do so for $\mathcal{J}_{(\mathrm{b})\mathrm{mn}}$ and $\mathcal{J}_{(\mathrm{f})\mathrm{mn}}$. $\mathcal{J}_{(\mathrm{f})\mathrm{mn}}$ is completely new as it is generated by the world line fermions and has no analogue in the bosonic SP matrix model. The 1D fermionic current $\mathcal{F}_{mn}$ can also be expanded as

$$\mathcal{F}_{mn} = \sum_{K=1}^{N^2} T^K \mathcal{F}_{nm}^K. \tag{3.14}$$

Note that $\mathcal{J}_{mn}$ and $\mathcal{F}_{mn}$ are not completely independent since they are related under the ST

$$\begin{aligned} \delta_\epsilon \mathcal{F}_{nm} &= \epsilon \mathcal{J}_{mn} \\ \delta_\epsilon \mathcal{J}_{nm} &= -i\epsilon \partial_t \mathcal{F}_{mn}\,. \end{aligned} \tag{3.15}$$

and (3.9). Their explicit component field expressions, defining the classical constraint equations, are given by

$$\mathcal{F}_{mn} = i\frac{B}{2}\varepsilon_{ab}\left[\eta^a, X^b\right]_{nm} + \frac{1}{2}\left(\upsilon_n^* \Psi_m + \Psi_m^* \upsilon_n\right) \tag{3.16}$$

$$\mathcal{J}_{(\mathrm{b})mn} = i\frac{B}{2}\varepsilon_{ab}\left[X^a, X^b\right]_{nm} + \Psi_n^* \Psi_m \tag{3.17}$$

$$\mathcal{J}_{(\mathrm{f})mn} = -\frac{B}{2}\varepsilon_{ab}\left\{\eta^a, \partial_t \eta^b\right\}_{nm} + \frac{i}{2}\left(\upsilon_n^* \partial_t \upsilon_m - \upsilon_n \partial_t \upsilon_m^*\right). \tag{3.18}$$

These quantities can be rewritten in terms of the U(1) $\oplus$ SU(N) generators in a fashion similar to (2.11)–(2.13). The new thing, which appears in comparison to the SP theory, is that these generators have two sectors in one-to-one correspondence with the particles and their super-partners. For instance, the total U(1) charge $\mathcal{J}^0 = \mathrm{Tr}(\mathcal{J}_{mn})$ has now two contributions $\mathcal{J}_{(\mathrm{b})mn} \equiv \mathcal{J}_\Psi^0$ and $\mathcal{J}_{(\mathrm{f})mn} \equiv \mathcal{J}_\upsilon^0$. The old

$$\mathcal{J}_\Psi^0 = N\left(\Psi^* Q \Psi\right) \tag{3.19}$$

is realized in terms of the Polychronakos bosonic field Ψ as in (3.17) and the extra contribution

$$\mathcal{J}_\upsilon^0 = \frac{i}{2} N\left(\upsilon^* Q \partial_t \upsilon - \upsilon Q \partial_t \upsilon^*\right) \tag{3.20}$$

comes from the world line υ fermions (3.18). Therefore, in addition to the SU(N) invariance, the constraint relation regarding the U(1) charge reads

$$\mathcal{J}^0 = \mathcal{J}_\Psi^0 + \mathcal{J}_\upsilon^0 = Nk \tag{3.21}$$

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where its two parts are

$$\mathcal{J}^0_\Psi = Nk_1\,, \qquad \mathcal{J}^0_v = Nk_2 \tag{3.22}$$

implying that

$$k_1 + k_2 = k\,. \tag{3.23}$$

With this splitting, one can compare at any step with what we know about the world line bosonic case and learn exactly what is the contribution of the world line fermions.

Quantum constraints : in this case the classical constraint equations (3.16)–(3.22) should be replaced by those for the Hilbert space of the SMM

$$\langle\mathcal{F}|\left[\mathcal{J}^K_\Psi + \mathcal{J}^K_v\right] T^K_{mn}|\mathcal{F}\rangle = 0 \tag{3.24}$$

$$\langle\mathcal{F}|\mathcal{J}^0|\mathcal{F}\rangle = \langle\mathcal{F}|\mathcal{J}^0_\Psi + \mathcal{J}^0_v|\mathcal{F}\rangle = Nk_1 + Nk_2 \tag{3.25}$$

$$\langle\mathcal{F}|\mathcal{F}^K T^K_{mn}|\mathcal{F}\rangle = 0 \tag{3.26}$$

where $K = 1,\ldots,N^2-1$ and the state $|\mathcal{F}\rangle$ is a generic wave function of the underlying Hilbert space $\mathcal{H}$ of the hamiltonian $\mathbf{H}_{\rm susy}$ of the world line SMM. As in the SP model, the corresponding $\mathbf{H}_{\rm susy}$ operator, which may also be written in terms of the total quantum flux operator as $\omega\mathcal{N}_\phi$ (2.39), plays a crucial role in the SMM. It has two operator contributions, one coming from the world line bosonic operators and the other from their superpartners. It reads

$$\mathbf{H} = \omega\left(\mathcal{N}_{\rm A} + \mathcal{N}_{\rm C} + \frac{a_{\rm b} + a_{\rm f}}{2}\right) \tag{3.27}$$

where

$$\mathcal{N}_{\rm A} = \sum_{n,m} \mathbf{A}^\dagger_{nm}\mathbf{A}^-_{nm}\,, \qquad \mathcal{N}_{\rm C} = \sum_{n,m} \mathbf{C}^\dagger_{nm}\mathbf{C}^-_{nm} \tag{3.28}$$

are, respectively, the number matrix operators counting the world line bosonic and fermionic excitations. The number $a_{\rm b} + a_{\rm f}$ is the total quantum correction generated by the ordering of the (world line bosonic and fermionic) creation and annihilation matrix operators $\mathbf{A}^\pm_{nm}$ and $\mathbf{C}^\pm_{nm}$. In general it is equal to

$$a_{\rm b} + a_{\rm f} = N_{\rm e}^2 - N_{\tilde{\rm e}}^2 \tag{3.29}$$

and cancels exactly due to the world line SUSY, which requires an equal number of bosonic and fermionic degrees of freedom

$$N_{\rm e} = N_{\tilde{\rm e}}\,. \tag{3.30}$$

There is nothing strange with this kind of cancellations. It is well-known in supersymmetric quantum field theories. But the novelty here is that in the SMM, one expects that there will be no Polychronakos quantum effect for the total filling factor, the bosonic and fermionic effects cancel each other. We will turn to this special feature later, for the moment let us build up the ground state of our model.

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3.2.2 Solving quantum constraints

We begin by giving some useful tools in order to derive the lowest-energy wave function of our model. As our system involves two kinds of operators, the corresponding Hilbert space is a product of the bosonic and fermionic sectors such as $\mathbb{F}_{\mathrm{b}} \times \mathbb{F}_{\mathrm{f}}$. They involve, respectively, the bosonic $(\mathbf{A}^{\dagger}_{nm}, \mathbf{A}^{-}_{mn})$ and fermionic $(\mathbf{C}^{\dagger}_{nm}, \mathbf{C}^{-}_{mn})$ operators acting on $|0\rangle$ and $|S\rangle$ vacua. Since the analysis of the first sector has been discussed in section 2, let us discuss below that dealing with quantum world line fermions.[4]

Fermionic oscillators: note that on a generic basis, the harmonic oscillator matrix operators $(\mathbf{C}^{\dagger}_{kl}, \mathbf{C}^{-}_{nm})$ form a system of N^2 fermionic (Ramond like sector) creation and annihilation operators. They are the superpartners of the world line bosonic operators $(\mathbf{A}^{\dagger}_{nm}, \mathbf{A}^{-}_{mn})$ of the SP model and satisfy the Clifford algebra

$$\begin{aligned} \left\{\mathbf{C}^{\dagger}_{kl}, \mathbf{C}^{-}_{nm}\right\} &= -i\delta_{kn}\delta_{lm} \\ \left\{\mathbf{C}^{\dagger}_{kl}, \mathbf{C}^{\dagger}_{nm}\right\} &= \left\{\mathbf{C}^{-}_{kl}, \mathbf{C}^{-}_{nm}\right\} = 0\,. \end{aligned} \tag{3.31}$$

where the different indices are running from 1 to N. It is known that this algebra has a spinor representation of dimension $2^{[\frac{N^2}{2}]}$. Therefore the ground state $|S\rangle$ of the fermionic field Fock space $\mathbb{F}_{\mathrm{f}}$ is a $2^{\frac{N^2}{2}}$ dimensional spinor in contrary to the bosonic Fock space $\mathbb{F}_{\mathrm{b}}$ where the vacuum $|0\rangle$ is 1D. Note that the degeneracy in $|S\rangle$ is not relevant from the world line SUSY view. It might be interesting in building a target space supersymmetric fractional QH matrix theory, but this generalization goes beyond the scope of the present study. The representations of (2.20) and (3.31) imply

$$\mathbf{C}^{-}_{nm}|S\rangle = 0\,, \qquad \mathbf{H}_{\mathrm{susy}}|S\rangle = -\frac{\omega}{2}N^2|S\rangle \tag{3.32}$$

$$\mathbf{A}^{-}_{nm}|0\rangle = 0\,, \qquad \mathbf{H}_{\mathrm{susy}}|0\rangle = \frac{\omega}{2}N^2|0\rangle \tag{3.33}$$

$$\mathbf{A}^{\dagger}_{mn}|0\rangle = |1\rangle\,, \qquad \mathbf{C}^{\dagger}_{mn}|S\rangle = |S^1\rangle \tag{3.34}$$

where $|S^1\rangle$ and $|1\rangle$ are the first excited states of the fermionic and bosonic oscillators. Their respective energies are $\frac{\omega}{2}(1-N^2)$ and $\frac{\omega}{2}(1+N^2)$. Generic states of the tensor product Hilbert space of the world line SMM are generated by the product of the following states

$$\prod_{i\geq 1}^{R_{\mathrm{f}}} \mathbf{C}^{\dagger}_{k_i l_i}|S\rangle\,, \qquad \prod_{i\geq 1}^{R_{\mathrm{b}}} \mathbf{A}^{\dagger}_{k_i l_i}|0\rangle \tag{3.35}$$

where R_{b} and R_{f} are positive integers. Of course the first excited states (3.34) are not the general ones and (3.35) cannot be solutions for the quantum supersymmetric constraints (3.24)–(3.25). The point is that these states are not SU(N) gauge invariant and do

[4] A priori, the quantization of world line fermions may have Neveu-Schwartz (NS) and Ramond (R) like sectors in analogy with two-dimensional conformal field theory. Here we focus on the R sector, but a complete analysis would involve all sectors. It is in this way that one may consider target space supersymmetric fractional QH models.

not have the appropriate U(1) group charge $k_1 N + k_2 N$. To find the appropriate solutions, we proceed as follows.

Extra restriction and D phase: note that R_{b} appearing in the generating monomials (3.35) can take any positive integer value, but R_{f} should not exceed N^2 because of the world line fermion statistics. Indeed from (3.31), one can see that the individual $\mathbf{C}_{kl}^{\dagger}$ operators are nilpotent

$$\left(\mathbf{C}_{kl}^{\dagger}\right)^2 = 0 \tag{3.36}$$

and therefore the non-zero wave functions should have no more than $\prod_{i,j=1}^{N} \mathbf{C}_{ij}^{\dagger}$ fermionic excitations. Otherwise they are nilpotent because

$$\mathbf{C}_{kl}^{\dagger} \left(\prod_{i,j=1}^{N} \mathbf{C}_{ij}^{\dagger} \right) = 0\,. \tag{3.37}$$

This means that the wave functions for our model, which we are looking for, should have no more than N^2 world line fermions, but can have any number of world line bosons. The following proposition summarizes this particular property and makes a link with the U and D phases.

Proposition: the U(N) gauge invariant ground state configuration of the world line SMM describing a set of N supermultiplets in an external magnetic field B has two phases:

(a) The world line boson phase U corresponds in standard fractional QH theory to that system described by the SP theory. Therefore the flux number is given by

$$N_{\phi}^{\mathrm{b}} = (k_1 + 1)\, N \tag{3.38}$$

leading to

$$\nu_{\mathrm{b}} = \frac{1}{k_1 + 1} \tag{3.39}$$

where k_1 is a (odd) positive integer. This result agrees with the Polychronakos effect.

(b) The world line fermion phase D has the flux number

$$N_{\phi}^{\mathrm{f}} = (1 - k_2)\, N \tag{3.40}$$

where k_2 restricted to the values 0 and 1 in agreement with the Pauli exclusion principle. The corresponding filling factor reads

$$\nu_{\mathrm{f}} = \frac{1}{1 - k_2}\,. \tag{3.41}$$

It shows that the quantum corrections generate an anti-Polychronakos effect. This is an interesting feature since it will play a crucial role in characterizing the QH system.

(*c*) From the above analysis, we conclude that the total filling factor $\nu_{\rm tot}$ of the supersymmetric fractional QH system is

$$\nu_{\rm tot} = \frac{\nu_{\rm b}\nu_{\rm f}}{\nu_{\rm b} - \nu_{\rm f}} \,. \tag{3.42}$$

It is clear that the Polychronakos and anti-Polychronakos effects are equal. Since they are with opposite sign, they cancel each other.

To prove this proposition, we need the ground state configuration. This can be obtained by solving the quantum supersymmetric constraints, which requires U(N) gauge invariant wave functions. Since we have U(N) = U(1) × SU(N), we will first solve the constraint equations for the U(1) subsymmetry and second those for SU(N).

U(1) *symmetry and extra oscillators*: recall that in the bosonic case, the U(1) gauge invariance is solved with the help of the Polychronakos vector operators $(\mathbf{\Psi}_i^\dagger, \mathbf{\Psi}_i^-)$ as shown in (2.26). The same thing happens for the supersymmetric case. Instead of the Polychronakos operators, the job will be done by the supersymmetric pairs $(\mathbf{\Psi}_i^\dagger, v_i^\dagger)$ and $(\mathbf{\Psi}_i^-, v_i^-)$. To see how things work, note that the general form of the wave functions in the full Hilbert space is generated, in addition to (3.35), by the extra monomials

$$\prod_{i\geq 1}^{r_{\rm f}} v_{k_i}^\dagger |s\rangle \,, \qquad \prod_{i\geq 1}^{r_{\rm b}} \mathbf{\Psi}_{k_i}^\dagger |0\rangle \tag{3.43}$$

where $r_{\rm f}$ and $r_{\rm b}$ are positive integers, the vacuum $|s\rangle$ will be specified later. To avoid confusion, let us rewrite the supersymmetric quantum constraints (3.24)–(3.26) into the equivalent forms

$$\langle\mathcal{F}|\mathcal{J}_\Psi^0|\mathcal{F}\rangle = Nk_1 \tag{3.44}$$

$$\langle\mathcal{F}|\mathcal{J}_{\rm (b)}^K T_{mn}^K|\mathcal{F}\rangle = 0 \tag{3.45}$$

$$\langle\mathcal{F}|\mathcal{J}_v^0|\mathcal{F}\rangle = Nk_2 \tag{3.46}$$

$$\langle\mathcal{F}|\mathcal{J}_{\rm (f)}^K T_{mn}^K|\mathcal{F}\rangle = 0 \tag{3.47}$$

$$\langle\mathcal{F}|\mathcal{F}^K T_{mn}^K|\mathcal{F}\rangle = 0 \tag{3.48}$$

and consider the solution of each of them.

Solving equations (3.44)–(3.45): The first equation tells us that the wave function $|\mathcal{F}\rangle$ should have a monomial of type $\prod_{i\geq 1}^{k_1} \mathbf{\Psi}_{k_i}^\dagger$. However the second requires that $|\mathcal{F}\rangle$ should be SU(N) invariant. The lowest-energy solution $|\mathcal{F}_{\rm b}^0\rangle$ fulfilling these two requirements is that given in (2.26)

$$|\mathcal{F}_{\rm b}^0\rangle = \mathcal{N}_1 \left[\varepsilon^{\alpha_1\cdots\alpha_N}\, \mathbf{\Psi}_{\alpha_1}^\dagger \left(\mathbf{\Psi}^\dagger\cdot\mathbf{A}^\dagger\right)_{\alpha_2} \cdots \left(\mathbf{\Psi}^\dagger\cdot\mathbf{A}^{\dagger N}\right)_{\alpha_N}\right]^{k_1} |0\rangle \tag{3.49}$$

where the index b refers to the bosonic part of the full state $|\mathcal{F}^0\rangle = |\mathcal{F}_{\rm b}^0\rangle \otimes |\mathcal{F}_{\rm f}^0\rangle$ and the upper index 0 to the lowest energy solution. By using (2.40) and (2.51), one can see that the bosonic contribution to the filling factor $\nu_{\rm b}$ of the state $|\mathcal{F}_{\rm b}^0\rangle$ coincides with that given by relation (3.39). This is exactly the expected value with the usual Polychronakos

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quantum shifting k_1 to k_1+1. Here there is no constraint of the (odd) positive integer k_1 and so the flux number $N_\phi^{(\mathrm{b})}$ may have any value. This establishes the features of the U phase of proposition (a).

Solving equations (3.46)–(3.47): note that the N superpartners $(\upsilon_n^\dagger, \upsilon_n^-)$ of the Polychronakos $(\mathbf{\Psi}_i^\dagger, \mathbf{\Psi}_i^-)$ operators satisfy a N-dimensional Clifford algebra

$$\begin{aligned}\left\{\upsilon_m^\dagger, \upsilon_n^-\right\} &= -i\delta_{mn}\\ \left\{\upsilon_m^\dagger, \upsilon_n^\dagger\right\} &= \{\upsilon_m^-, \upsilon_n^-\} = 0\\ \left[\mathcal{J}_\upsilon^0, \upsilon_n^\pm\right] &= \pm\upsilon_n^\pm\,. \end{aligned}\tag{3.50}$$

In these relations

$$\mathcal{J}_\upsilon^0 = \sum_n i\upsilon_n^\dagger \upsilon_n^- \tag{3.51}$$

is the fermionic part of the U(1) current of (3.21)–(3.22). It should be compared with the bosonic term

$$\mathcal{J}_{\mathbf{\Psi}}^0 = \sum_{n=1}^N \mathbf{\Psi}_n^\dagger \mathbf{\Psi}_n^- \tag{3.52}$$

encountered in section 2. From (3.50), we learn that $\upsilon_n^\pm$ can be realized as $2^{[\frac{N}{2}]}\times 2^{[\frac{N}{2}]}$ matrices acting on $2^{[\frac{N}{2}]}$ dimensional spinors $|s_\gamma\rangle$

$$\upsilon_n^- |s_\gamma\rangle = 0 \tag{3.53}$$

$$\mathcal{J}_\upsilon^0 |s_\gamma\rangle = 0 \tag{3.54}$$

$$\upsilon^\dagger |s_\gamma\rangle = |1, s_\gamma\rangle\,. \tag{3.55}$$

The vacuum state $|S\rangle$ of the $\mathbf{C}_{kl}^\pm$ Clifford algebra can be realized in terms of the above $|s_\gamma\rangle$ by taking tensor products

$$|S\rangle = \prod_{i=1}^N |s_{\gamma_i}\rangle\,. \tag{3.56}$$

Since in the algebras (3.31), (3.50) both sets of $\mathbf{C}_{kl}^\dagger$ and $\upsilon_i^\dagger$ operators are nilpotent

$$\left(\mathbf{C}_{kl}^\dagger\right)^2 = 0\,, \qquad \left(\upsilon_i^\dagger\right)^2 = 0 \tag{3.57}$$

one has extra constraints, which should be taken into account in building up the fermionic contribution $|\mathcal{F}_\mathrm{f}^0\rangle$. Due to the nilpotency property (3.37) as well as

$$\upsilon_i^\dagger \prod_{j=1}^N \upsilon_j^\dagger = 0 \tag{3.58}$$

$|\mathcal{F}_\mathrm{f}^0\rangle$ should have no more than N^2 creation operators $\mathbf{C}_{kl}^\dagger$ and no more than N operators $\upsilon_i^\dagger$. This feature means a strong constraint in constructing the solution of (3.44)–(3.48). It should be linked with the D phase of the SMM, i.e. proposition (b). With these tools, it is not difficult to see that the low-energy gauge invariant solution satisfying (3.46)–(3.47) is

$$|\mathcal{F}^0_{\mathrm{f}}\rangle = \mathcal{N}_2 \left[\varepsilon^{i_1\cdots i_N} \upsilon^\dagger_{i_1} \left(\upsilon^\dagger \cdot \mathbf{C}^\dagger\right)_{i_2} \cdots \left(\upsilon^\dagger \cdot \mathbf{C}^{\dagger N}\right)_{i_N}\right]^{k_2} |S\rangle \tag{3.59}$$

where the involved products are

$$\upsilon^\dagger \cdot \mathbf{C}^\dagger_i = \sum_j \upsilon^\dagger_j \cdot \mathbf{C}^\dagger_{ji}\,, \qquad \upsilon^\dagger \cdot \mathbf{C}^{\dagger 2}_i = \sum_{j,k} \upsilon^\dagger_k \cdot \mathbf{C}^\dagger_{kj}\mathbf{C}^\dagger_{ji} \tag{3.60}$$

and so on. In this relation the charge k_2 seems to play a role completely equivalent to the bosonic charge k_1. However this is not true because of the nilpotency relations (3.37), (3.58), which require that k_2 may take only two values

$$k_2 = 0, 1\,. \tag{3.61}$$

This restriction is a manifestation of the Pauli exclusion principle for fermions. But it has several consequences, which can be listed as follows.

(i) The flux number $N^{(\mathrm{f})}_\phi = (1 - k_2)N$ is bounded. This shows that the D phase has a filling factor ν_{f} a priori given by (3.41).

(ii) We distinguish two different solutions:

Case $k_2 = 0$: here the lowest energy configuration is very special since it corresponds to the vacuum $|S\rangle$, which is a wave function without $\upsilon^\dagger_i$ and $\mathbf{C}_{ij}{}^\dagger$ dependence. Its vacuum energy is given by the classical null value shifted by the negative quantum correction

$$-\frac{\omega N^2}{2}\,. \tag{3.62}$$

This is a remarkable feature, namely world line fermions induce a negative quantum flux number

$$(k_2 - 1)N = -N^{(\mathrm{f})}_\phi\,. \tag{3.63}$$

Now by defining $N^{(\mathrm{f})}_\phi$ as the opposite of this number

$$N^{(\mathrm{f})}_\phi = (1 - k_2)N = 1 \tag{3.64}$$

we obtain an integer filling factor for the D phase. On the other hand, following (2.42), (2.58) we have

$$N_\phi = k_1 N + (N - N) = k_1 N \tag{3.65}$$

where two quantum corrections cancel each other exactly. Then the total filling factor ν_{tot} of the ground state $|\mathcal{F}^0\rangle$ is given by

$$\nu_{\mathrm{tot}} = \frac{1}{k_1} = \frac{1}{k}\,. \tag{3.66}$$

Using the relation (3.65), one can see that (3.66) can also be rewritten as (2.58). This establishes the point (c) of our proposition for $k_2 = 0$.

Case $k_2 = 1$: in this case, the ground state $|\mathcal{F}^0\rangle$ of the super fractional QH system depends on the operators $\upsilon_i^\dagger$ and $\mathbf{C}_{kl}^\dagger$. Therefore the resulting state can be written as

$$|\mathcal{F}^0\rangle = \mathcal{N}\left[\varepsilon^{\alpha_1\cdots\alpha_N}\Psi_{\alpha_1}^\dagger\left(\Psi^\dagger\cdot\mathbf{A}^\dagger\right)_{\alpha_2}\cdots\left(\Psi^\dagger\cdot\mathbf{A}^{\dagger N}\right)_{\alpha_N}\right]^{k_1}\times$$
$$\times\varepsilon^{i_1\cdots i_N}\upsilon_{i_1}^\dagger\left(\upsilon^\dagger\cdot\mathbf{C}^\dagger\right)_{i_2}\cdots\left(\upsilon^\dagger\cdot\mathbf{C}^{\dagger N}\right)_{i_N}|0\rangle\otimes|S\rangle\,. \tag{3.67}$$

The total number M of $\upsilon_i^\dagger$ and $\mathbf{C}_{ij}^\dagger$ is equal to

$$M = \frac{N(N+3)}{2}\,. \tag{3.68}$$

From the world line SUSY view, this state behaves as a bosonic state if M is an even integer number and as a fermion if M is odd. Now using (2.58) and the energies

$$\begin{aligned} E_{\mathrm{b}} &= \frac{\omega}{2}[k_1 N(N+1)+N^2] \\ E_{\mathrm{f}} &= \frac{\omega}{2}[N(N+1)-N^2] = \frac{\omega}{2}N \end{aligned} \tag{3.69}$$

we can show that the total filling factor corresponding to (3.67) is

$$\nu_{\mathrm{tot}} = \frac{1}{k_1+1} = \frac{1}{k}\,. \tag{3.70}$$

Here also the quantum corrections coming from bosons and fermions cancel exactly due to the world line SUSY. This tells us that Polychronakos effects cancel exactly in supersymmetric FQH systems.

3.3 Superfield matrix fractional QH model

Our aim here is to show that, in general, it is more convenient to adopt the superspace method in the study of our SMM. This is a manifestly supersymmetric formulation which allows a deeper insight into the general properties of the fractional QH system without entering into details about bosonic and fermionic contributions. It allows, amongst others, a derivation of direct results extending of the non-commutative Susskind theory for the Laughlin fluid [1]. Here we will mainly focus on building the superfield extension of the SP model and make a link with the previous component field analysis.

3.3.1 $\mathcal{N} = 1$ worldline SUSY

The SMM of the fractional QH system, involving $N_{\mathrm{e}} + N_{\tilde{\mathrm{e}}}$ particles and superpartners, has a supersymmetric conserved charge Q. This is related to the usual conserved hamiltonian $\mathbf{H}_{\mathrm{susy}}$ of the fractional QH system by

$$Q^2 = \mathbf{H}_{\mathrm{susy}}\,. \tag{3.71}$$

This defines a 1D supersymmetric algebra and has irreducible linear representations $\mathcal{R}_s$, which are completely characterized by a superspin quantum number s, with $2s$ integer.

In supersymmetric 1D field formulation, a convenient way to realize $\mathcal{R}_s$ is to work in the superspace (t, ζ) and use the supersymmetric covariant derivative D

$$DQ = -QD \tag{3.72}$$

satisfying the superalgebra

$$D^2 = E = i\frac{\partial}{\partial t}\,. \tag{3.73}$$

In the (t, ζ) superspace, D and E generators are related by

$$D = \frac{\partial}{\partial \zeta} + \zeta E = \frac{\partial}{\partial \zeta} + i\zeta\frac{\partial}{\partial t} \tag{3.74}$$

and $\mathcal{R}_s$ are described by 1D superfields Φ_s. These can be expanded as

$$\Phi_s(t, \zeta) = \varphi_s(t) + \zeta\xi_{s+\frac{1}{2}}(t)\,. \tag{3.75}$$

The scale dimensions of the component fields and their functionals may be directly obtained by using the scales of ζ and t, such as ζ, t, D and ∂_t carry scale values $-1/2$, -1, $1/2$ and 1, respectively. For the special example where $s = -\frac{1}{2}$, which is the case when the component field $\xi_{s+\frac{1}{2}}(t)$ is a 1D scalar, the SE of the term $\xi^1\partial_t\xi^2$ is obtained in steps, by computing first

$$\begin{aligned} D\Phi_s &= \xi_{s+\frac{1}{2}} + i\zeta\partial_t\varphi_s \\ \partial_t\Phi_s &= \partial_t\varphi_s + \zeta\partial_t\xi_{s+\frac{1}{2}} \end{aligned} \tag{3.76}$$

and by looking second for the appropriate combination that contains $\xi^1\partial_t\xi^2$. Then the SE of $\xi^1\partial_t\xi^2$ is

$$\int d\zeta\, D\Phi^1\partial_t\Phi^2 = \xi^1\partial_t\xi^2 + \partial_t\varphi^1\partial_t\varphi^2\,. \tag{3.77}$$

In similar fashion we can compute others supersymmetric invariants and in particular

$$\int d\zeta\Phi D\Phi = \xi\xi + \varphi\partial_t\varphi\,. \tag{3.78}$$

In the presence of the gauge symmetries, D and ∂_t should be replaced by the gauge covariant derivatives

$$\begin{aligned} \mathcal{D} &= D - i\mathcal{A} = (\partial_\zeta - i\alpha) + i\zeta\,(\partial_t - i\gamma) \\ \nabla_t &= \nabla = \partial_t - i\mathcal{C} = \partial_t - iA + i\zeta\rho \end{aligned} \tag{3.79}$$

where $\mathcal{A}$ and $\mathcal{C}$ are superfields related to each other by the constraint equations

$$\begin{aligned} \{\mathcal{D}, \mathcal{D}\} &= 2\mathcal{D}^2 = 2i\nabla \\ [\mathcal{D}, \nabla] &= 0\,. \end{aligned} \tag{3.80}$$

These equations imply, in general, that $\mathcal{A}$ and $\mathcal{C}$ are not really independent since they are connected by

$$D\mathcal{A} + \frac{i}{2}\{\mathcal{A}, \mathcal{A}\} = \mathcal{C}\,. \tag{3.81}$$

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Also we recall the useful property

$$\int d\zeta \Phi = \mathcal{D}\Phi|_{\zeta=0} \tag{3.82}$$

for explicit computations. Having given the main superspace tools, now we turn to study the degrees of freedom, which we will use to build the supersymmetric generalization of the SP theory.

3.3.2 Degrees of freedom

In addition to the superspin described above, the matrix model for the fractional QH system involves extra quantum numbers associated with the symmetry group $G = \mathrm{SO}(2) \times \mathrm{U}(N)$. Here SO(2) is the symmetry group of the $\mathbb{R}^2$ plane and $\mathrm{U}(N)$ the usual gauge group of the SP model. We need the superfields that transform under the representations $(\mathbf{r}, \mathrm{R})$ of $\mathrm{SO}(2) \times \mathrm{U}(N)$ and have the ζ-expansions

$$\Phi_{(\mathbf{r},\mathrm{R})} = \varphi_{(\mathbf{r},\mathrm{R})} + \zeta \xi_{(\mathbf{r},\mathrm{R})} \,. \tag{3.83}$$

For simplicity, we have dropped the superspin index. The superfield contents of our SMM are

Fields	χ^a	Φ	$\mathcal{A}$	$\mathcal{C}$	Γ
Superspin s	$-1/2$	$-1/2$	$1/2$	1	$-1/2$
SO(2) repres. $\mathbf{r}$	2	1	1	1	1
$\mathrm{U}(N)$ repres. R	$\mathbf{N}\times\overline{\mathbf{N}}$	$\mathbf{N}$	$\mathbf{N}\times\overline{\mathbf{N}}$	$\mathbf{N}\times\overline{\mathbf{N}}$	$\mathbf{N}\times\overline{\mathbf{N}}$

(3.84)

where χ^a parameterizes superpositions and supermomentum of the supersymmetric particles, Φ is the SE of the Polychronakos field. $\mathcal{A}$ and $\mathcal{C}$ are the supersymmetric gauge fields needed to capture the constraint equations on the super fractional QH system. Γ is a fermionic superfield carrying the supersymmetric constraint equation (3.80). In terms of the components fields (η^a, X^a), (υ, Ψ), (β, b), (α, γ) and (A, ρ) of χ^a, Φ, Γ, $\mathcal{A}$ and $\mathcal{C}$ respectively, the previous table becomes

Fields	$\eta^a = \chi^a\|$	$\mathrm{i}X^a = \mathcal{D}\chi^a\|$	$\upsilon = \Phi\|$	$\mathrm{i}\Psi = \mathcal{D}\Phi\|$	$\alpha = \mathcal{A}\|$	$\mathrm{i}\gamma = \mathcal{D}\mathcal{A}\|$
Superspin s	$-1/2$	0	$-1/2$	0	$1/2$	1
SO(2) repres. $\mathbf{r}$	2	2	1	1	1	1
$\mathrm{U}(N)$ repres. R	$\mathbf{N}\times\overline{\mathbf{N}}$	$\mathbf{N}\times\overline{\mathbf{N}}$	$\mathbf{N}$	$\mathbf{N}$	$\mathbf{N}\times\overline{\mathbf{N}}$	$\mathbf{N}\times\overline{\mathbf{N}}$

(3.85)

and

Fields	$A = \mathcal{C}\|$	$\mathrm{i}\rho = \mathcal{D}\Gamma\|$	$\beta = \Gamma\|$	$\mathrm{ib} = \mathcal{D}\Gamma\|$
Superspin s	1	$3/2$	$-1/2$	0
SO(2) repres. $\mathbf{r}$	1	1	1	1
$\mathrm{U}(N)$ repres. R	$\mathbf{N}\times\overline{\mathbf{N}}$	$\mathbf{N}\times\overline{\mathbf{N}}$	$\mathbf{N}\times\overline{\mathbf{N}}$	$\mathbf{N}\times\overline{\mathbf{N}}$

(3.86)

where $\chi^a|$ stands for $\chi^a|_{\zeta=0}$ and so on.

3.3.3 Superfield constraints

One way to construct the superfield constraint equations is to start from the bosonic constraints and use the ST of the component fields

$$\begin{aligned} \delta_\epsilon \eta^a &= \epsilon X^a\,, & \delta_\epsilon X^a &= -i\epsilon\partial_t \eta^a \\ \delta_\epsilon \upsilon &= \epsilon \Psi\,, & \delta_\epsilon \Psi &= -i\epsilon\partial_t \upsilon \\ \delta_\epsilon \zeta &= \epsilon \end{aligned} \tag{3.87}$$

together with analogous relations for the remaining fields. In doing so, one can already derive partial information on the supersymmetrization of the U(N) constraint equations (2.3), which by the help of (3.15) can be written as

$$F_{mn} = i\frac{B}{2}\varepsilon_{ab}\left[\eta^a, X^b\right]_{nm} + \upsilon_n^*\Psi_m = \zeta B\theta\delta_{nm} \tag{3.88}$$

$$J_{mn} = i\frac{B}{2}\varepsilon_{ab}\epsilon\left(\left[X^a, X^b\right]_{nm} + i\left\{\,\eta^a, \partial_t\eta^b\right\}_{nm}\right) + \epsilon\Psi_n^*\Psi_m + i\epsilon\upsilon_n^*\partial_t\upsilon_m = \epsilon B\theta\delta_{nm} \tag{3.89}$$

where ζ is the superspace odd coordinate. As we have done before, it is convenient to split these constraints into the irreducible SU(N) and U(1) parts

$$\begin{aligned} \mathcal{F}_{mn} &= F_{mn} - \frac{1}{N}\mathcal{F}^0 \\ \mathcal{J}_{mn} &= J_{mn} - \frac{1}{N}\mathcal{J}^0 \end{aligned} \tag{3.90}$$

where the component field expressions for $\mathcal{F}^0$ and $\mathcal{J}^0$ are given by

$$\mathcal{F}^0 = \sum_{n=1}^{N} \upsilon_n^*\Psi_n + \Psi_n\upsilon_n^* = 2B\theta N\zeta \tag{3.91}$$

$$\mathcal{J}^0 = \sum_{n=1}^{N} \Psi_n^*\Psi_n + \Psi_n\Psi_n^* + i\left(\upsilon_n^*\partial_t\upsilon_n - \partial_t\upsilon_n\upsilon_n^*\right) = 2B\theta N\,. \tag{3.92}$$

While (3.89), (3.92) are adequately defined, the relations (3.88), (3.91) break explicitly the SUSY. This difficulty is due to a supergauge fixing and may be surmounted by working in superspace and introducing auxiliary superfields. Let us describe briefly how the machinery works.

Supersymmetric action: in terms of the superfields of table (3.84), the simplest supersymmetric action $S[\chi^a, \Phi, \mathcal{C}, \mathcal{A}, \Gamma]$, which is U($N$) gauge invariant and generalizes the SP matrix model action (2.1) is

$$\begin{aligned} S\left[\chi^a, \Phi, \mathcal{C}, \mathcal{A}, \Gamma\right] = &-\frac{B}{2}\int dt\, d\zeta \left\{\mathrm{Tr}\left(\varepsilon_{ab}\chi^a\nabla\mathcal{D}\chi^b + \omega_0\chi^a\mathcal{D}\chi^a\right) + \theta\,\mathrm{Tr}\,\mathcal{A}\right\} + \\ &+\int dt\, d\zeta\left\{\sum_{n=1}^{N} i\Phi_n^*\nabla\mathcal{D}\Phi_n + \mathrm{Tr}\left[\Gamma\left(2D\mathcal{A} - i\left\{\mathcal{A},\mathcal{A}\right\} - 2\mathcal{C}\right)\right]\right\} \end{aligned} \tag{3.93}$$

where the first and the fourth terms are, respectively, the supersymmetrization of the Susskind and Polychronakos terms. The second term is a confining superpotential and

Γ is an auxiliary superfield carrying (3.80). The covariant derivatives $\mathcal{D}$ and ∇ act on superfields Υ in the U(N) adjoint as

$$\begin{aligned}\mathcal{D}\Upsilon &= \partial_\zeta \Upsilon - i\,[\mathcal{A},\Upsilon\} \\ \nabla\Upsilon &= \partial_t \Upsilon - i\,[\mathcal{C},\Upsilon]\end{aligned} \tag{3.94}$$

and on superfields in the fundamentals

$$\begin{aligned}\mathcal{D}\Phi &= \partial_\zeta \Phi - i\mathcal{A}\Phi \\ \nabla\Phi &= \partial_t \Phi - i\mathcal{C}\Phi\,.\end{aligned} \tag{3.95}$$

Before expanding the superfield action (3.93) in terms of the component fields, it is convenient to rewrite the above superfield action by using the complex superfield

$$\chi = \frac{1}{\sqrt{B}}\left(\chi^1 + i\chi^2\right) = \eta + \zeta Z \tag{3.96}$$

where the component fields are given by

$$\eta = \frac{1}{\sqrt{B}}\left(\eta^1 + i\eta^2\right) \tag{3.97}$$

and Z is given in subsection 2.1. In terms of this complex superfield, (3.93) takes the form

$$\begin{aligned}S = &\int dt\, d\zeta\, \mathrm{Tr}\left[\frac{i}{4}\left(\overline{\chi}\nabla\mathcal{D}\chi - \overline{(\nabla\mathcal{D}\chi)}\chi\right) - \frac{\omega_0}{4}\left(\overline{\chi}\mathcal{D}\chi + \overline{(\mathcal{D}\chi)}\chi\right)\right] + \\ &+ \int dt\, d\zeta \left\{\sum_{n=1}^{N}\frac{i}{2}\left(\overline{\Phi}_n\nabla\mathcal{D}\Phi_n - \overline{\nabla\mathcal{D}\Phi}_n\Phi_n\right) + \mathrm{Tr}\left[\Gamma\left(2D\mathcal{A} - i\left\{\mathcal{A},\mathcal{A}\right\} - 2\mathcal{C}\right)\right]\right\}.\end{aligned} \tag{3.98}$$

It can be further simplified by adding total superderivatives, a property which we will often use in the coming computations. From this action, one can compute the various superfield equations of motion. For dynamical χ and Φ superfields, we have

$$\begin{aligned}\nabla\mathcal{D}\chi + i\omega_0\mathcal{D}\chi &= 0 \\ \nabla\mathcal{D}\Phi &= 0\,.\end{aligned} \tag{3.99}$$

Using the ζ expansion (3.96), we can also derive the component field equations of motion for X, η, υ and Ψ. By using (3.96) as

$$\nabla Z + i\omega_0 Z = 0 \tag{3.100}$$

$$\nabla^2\eta + i\omega_0\nabla\eta = 0 \tag{3.101}$$

$$\nabla\Psi = 0 \tag{3.102}$$

$$\nabla^2\upsilon = 0\,. \tag{3.103}$$

Note that the superfield action as well as the various equations of motion are invariant under the U(N) gauge symmetry

$$\chi \to e^{i\Lambda}\chi e^{-i\Lambda}\,, \qquad \Phi \to e^{i\Lambda}\Phi \tag{3.104}$$

where

$$\Lambda(t) = \lambda(t) + \zeta\kappa(t) \tag{3.105}$$

is an arbitrary hermitian matrix gauge superfield parameter of U(N).

Superconstraints: to obtain the superfield constraint equations, we can minimize the action with respect to the nondynamical superfields Γ, $\mathcal{C}$ and $\mathcal{A}$. Using the identities

$$\begin{aligned}
\mathrm{Tr}\,\overline{\chi}\,[\nabla,\mathcal{D}\chi] &= \mathrm{Tr}\,\nabla\,[\mathcal{D}\chi,\overline{\chi}] \\
\mathrm{Tr}\,\overline{\chi}\,\{\mathcal{D},\chi\} &= \mathrm{Tr}\,\mathcal{D}\,\{\overline{\chi},\chi\} \\
\mathrm{Tr}\,\overline{\chi}\,\{\mathcal{D},\nabla\chi\} &= \mathrm{Tr}\,\mathcal{D}\,\{\overline{\chi},\nabla\chi\}
\end{aligned} \tag{3.106}$$

and the cyclic property of the trace as well as the statistics of the superfields, we find the equations of motion for the remaining superfields Γ, $\mathcal{C}$ and $\mathcal{A}$

$$2D\mathcal{A} - i\,\{\mathcal{A},\mathcal{A}\} - 2\mathcal{C} = 0 \tag{3.107}$$

$$\frac{1}{2}\,[\mathcal{D}\chi,\overline{\chi}] + \mathcal{D}\Phi\overline{\Phi} - 2\Gamma = 0 \tag{3.108}$$

$$i\frac{B}{2}\varepsilon_{ab}\left[\chi^a,\mathcal{D}\chi^b\right] + \Phi^*\mathcal{D}\Phi - 2\Gamma = 0 \tag{3.109}$$

$$\frac{1}{2}\,\{\overline{\chi},\nabla\chi\} + i\frac{\omega_0}{2}\,\{\overline{\chi},\chi\} + \nabla\Phi\overline{\Phi} - 2\mathcal{D}\Gamma = B\theta \tag{3.110}$$

$$-\frac{B}{2}\left(\varepsilon_{ab}\left\{\chi^a,\nabla\chi^b\right\} + \omega_0\,\{\chi^a\chi^a\}\right) + i\Phi^*\nabla\Phi - 2\mathcal{D}\Gamma = 2B\theta I\,. \tag{3.111}$$

Let us comment the contents of these relations. (3.107) contains two component field terms. It is a standard super relation which relates the various degrees of freedom of the components of the gauge superfields $\mathcal{A}$ and $\mathcal{C}$. It splits into

$$\begin{aligned}
\gamma - i\alpha^2 - A &= 0 \\
i\nabla_t\alpha &= \rho
\end{aligned} \tag{3.112}$$

where now $\nabla_t = \nabla| = \partial_t - i[A,\,]$ for the adjoint matrix fields. Note that in the special gauge $\alpha = \rho = 0$, the gauge field γ appearing in the expansion of $\mathcal{A}$ is just the gauge potential involved in $\mathcal{C}$. (3.108) gives the explicit expression of the superfield Γ in terms of χ and Φ. Its expansion in component fields leads to the constraints

$$i\frac{B}{2}\varepsilon_{ab}\left[\eta^a,X^b\right] + \upsilon^*\Psi - 2\beta = 0 \tag{3.113}$$

$$-\frac{iB}{2}\varepsilon_{ab}\left(\left[X^a,X^b\right] + i\left\{\eta^a,\nabla_t\eta^b\right\}\right) - \Psi^*\Psi - i\upsilon^*\nabla\upsilon - 2ib = 0\,. \tag{3.114}$$

Upon setting

$$A = 0\,, \qquad 2\beta = \zeta B\theta\,, \qquad 2ib = -B\theta \tag{3.115}$$

one discovers exactly (3.88), (3.89). Furthermore substituting (3.108) into (3.111), we obtain

$$-i\frac{B}{2}\varepsilon_{ab}\left[\mathcal{D}\chi^a,\mathcal{D}\chi^b\right] - B\varepsilon_{ab}\left\{\chi^a,\nabla\chi^b\right\} - \frac{\omega_0}{2}\,\{\chi^a,\chi^a\} - \mathcal{D}\Phi^*\mathcal{D}\Phi + 2i\Phi^*\nabla\Phi = \theta BI\,. \tag{3.116}$$

The first component field projection of the above superfield equation is

$$i\frac{B}{2}\varepsilon_{ab}\left[X^a,X^b\right] - B\varepsilon_{ab}\left\{\eta^a,\nabla\eta^b\right\} - \frac{\omega_0}{2}\,\{\eta^a,\eta^a\} + \Psi^*\Psi + 2i\upsilon^*\nabla\upsilon = \theta BI\,. \tag{3.117}$$

It reduces to (3.89) by using the equation of motion for A and setting $A = 0$. We now make some simplifications by working in the remarkable gauge $\alpha = 0$.

3.3.4 Component field action in gauge $\alpha = 0$

The supersymmetric action can be simplified by fixing a gauge such as $\alpha = 0$. This can be done by expanding the superfields in terms of the component fields

$$\chi = \eta + \zeta Z\,, \qquad \Phi = \upsilon + \zeta\Psi\,, \tag{3.118}$$

putting them into (3.93) and using the supersymmetric algebra $\mathcal{D}^2 = i\nabla$, by taking into account the statistics of the superfields and the integral measure. We find, up to total time derivatives, the component fields action

$$S\left[X,\eta,\Psi,\upsilon,A\right] = \int dt\,\mathcal{L} \tag{3.119}$$

where the lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \mathrm{Tr}\left[\frac{i}{2}\overline{Z}\nabla Z - \frac{1}{2}\left(\overline{\nabla\eta}\right)\nabla\eta - \frac{\omega_0}{2}\left(\overline{Z}Z - i\overline{\eta}\nabla\eta\right)\right] + i\overline{\Psi}\nabla\Psi - \overline{\nabla\upsilon}\nabla\upsilon + B\theta\,\mathrm{Tr}\,A \tag{3.120}$$

where A is now the unique auxiliary field. In this case the gauge covariant derivatives are defined by

$$\begin{aligned}
\nabla_t Z_{nm} &= \partial_t Z_{nm} - i\left[A,Z\right]_{nm}\\
\nabla_t \eta_{nm} &= \partial_t \eta_{nm} - i\left[A,\eta\right]_{nm}\\
\nabla_t \Psi_n &= \partial_t \Psi_n - iA_{nm}\Psi_m\\
\nabla_t \upsilon_n &= \partial_t \upsilon_n - iA_{nm}\upsilon_m\,.
\end{aligned}$$

Using the identities

$$\begin{aligned}
\mathrm{Tr}\left(\left[A,X^a\right]X^b\right) &= \mathrm{Tr}\left(A\left[X^b,X^a\right]\right)\\
\mathrm{Tr}\left(\left[A,\eta^a\right]\eta^b\right) &= \mathrm{Tr}\left(A\left\{\eta^a,\eta^b\right\}\right)
\end{aligned} \tag{3.121}$$

one can calculate the equation of motion for A. We find

$$\frac{1}{2}\left[Z,\overline{Z}\right] + i\left\{\overline{\eta},\nabla\eta\right\} + \frac{\omega_0}{2}\left\{\overline{\eta},\eta\right\} + \Psi\Psi^* - 2\nabla\upsilon\upsilon^* = B\theta\,. \tag{3.122}$$

This relation should be compared with (3.16), (3.17), (3.18). With this link, one can build the lowest energy configuration by solving the constraint equations in a similar fashion to those we have done in subsection 3.2.

4. Conclusion

We have developed a supersymmetric extension of the Susskind-Polychronakos matrix model. This is done by making use of component field and superfield methods. The one-dimensional $N \times N$ hermitian matrix field X^i and its superpartner η^i are involved. Our model has a supersymmetric U(N) gauge invariant action (3.12), (3.93), (3.98) containing the SP model. It is proposed to describe in this way a class of fractional QH systems. Our results can be summarized as follows:

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1. We have shown that the bosonic X^i and fermionic fields η^i, which are rotated under world line supersymmetry, have a gauge invariant vacuum state. This describes the lowest Landau level ground state with two phases U and D. (a) U is filled by bosonic excitations with no restriction on the flux number $N_\phi = k_1 N$ since k_1 can be any positive (odd) integer. (b) D contains world line fermions with flux number $N_\phi = k_2 N$ with $k_2 = 0, 1$. Since the filling of U and D is governed by the statistics of the quantum excitations, it is interesting to note that the similarity between the present analysis and Bose-Fermi phases of the quantum approximation for the ideal gas. In this approach, the U phase of our super matrix model for the fractional QH droplet can be seen as a Bose-like condensed state but with a filling constrained by U(N) gauge invariance. D is associated with the Fermi surface of the Fermi gas. Like in U, here also the U(N) gauge invariance puts a constraint on the radius of the Fermi surface.

2. We have constructed the ground state of the supersymmetric fractional QH system and shown that there is no Polychronakos shift for the filling factor in the supersymmetric case, because in our SE, there are two contributions to the Polychronakos effect. The first one is generated from bosonic excitations and the second one sums from fermionic excitations. These contributions are equal but with opposite signs, fermions generate an anti-Polychronakos effect. This exact cancellation is due to the world line supersymmetry.

3. We have also given the general relation between the total energy E of the system and the filling factor ν
$$\nu = \frac{\omega N^2}{2E} = \nu_0 + O\left(\frac{1}{N}\right). \tag{4.1}$$
This is used to study $1/N$ corrections in fractional QH systems and in particular allowed us to derive a description of a Hall system combining bosons and fermions.

Of course still some important questions remain to be answered, e.g., about the fractional charge and the statistics of the particles and how to describe them in terms of the proposed model. Another interesting question is related to the link between our model and super-Calogero [17] and super-Calogero-Sutherland [18] models. We will return to these issues and related matter in future.

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